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NAVAL POSTGRADUATE SCHOOL

Monterey, California

A PROCEDURE FOR ESTIMATING AN
OBJECT'S POSITION BASED ON TWO
OR MORE BEARINGS WITH A PROGRAM
FOR A TI-59 CALCULATOR

by

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)
The Report provides a procedure for estimating an object's position based on bearings taken from or on the object for two or more stations. The Report also provides a program for the TI-59 calculator to implement the procedure.

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A PROCEDURE FOR ESTIMATING AN OBJECT'S POSITION BASED ON TWO OR MORE BEARINGS WITH A PROGRAM FOR A TI-59 CALCULATOR

I. Introduction

A procedure for estimating an object's position with bearings taken on or from two or more stations is developed in Section IV of this report. In the development of the procedure, the following things are assumed: The object and the stations are fixed on the surface of a flat earth and the position of each station is known. The error in the bearing taken on or from a station is a normal random variable with a known standard deviation e and a mean of zero (if bias exists, it is known and removed); and station bearing errors are independent. The user instructions for a TI-59 program to implement the procedure are given in Section II, and the program listing is given in Section III.

As an example to illustrate a use of the program, suppose bearings are taken on an object from three stations (1, 2 and 3) as illustrated in Figure 1. Also, suppose that the assumptions stated above are satisfied and that an initial estimate of the object's position is made and that it is relatively near the object. This assumption is discussed in Section IV.

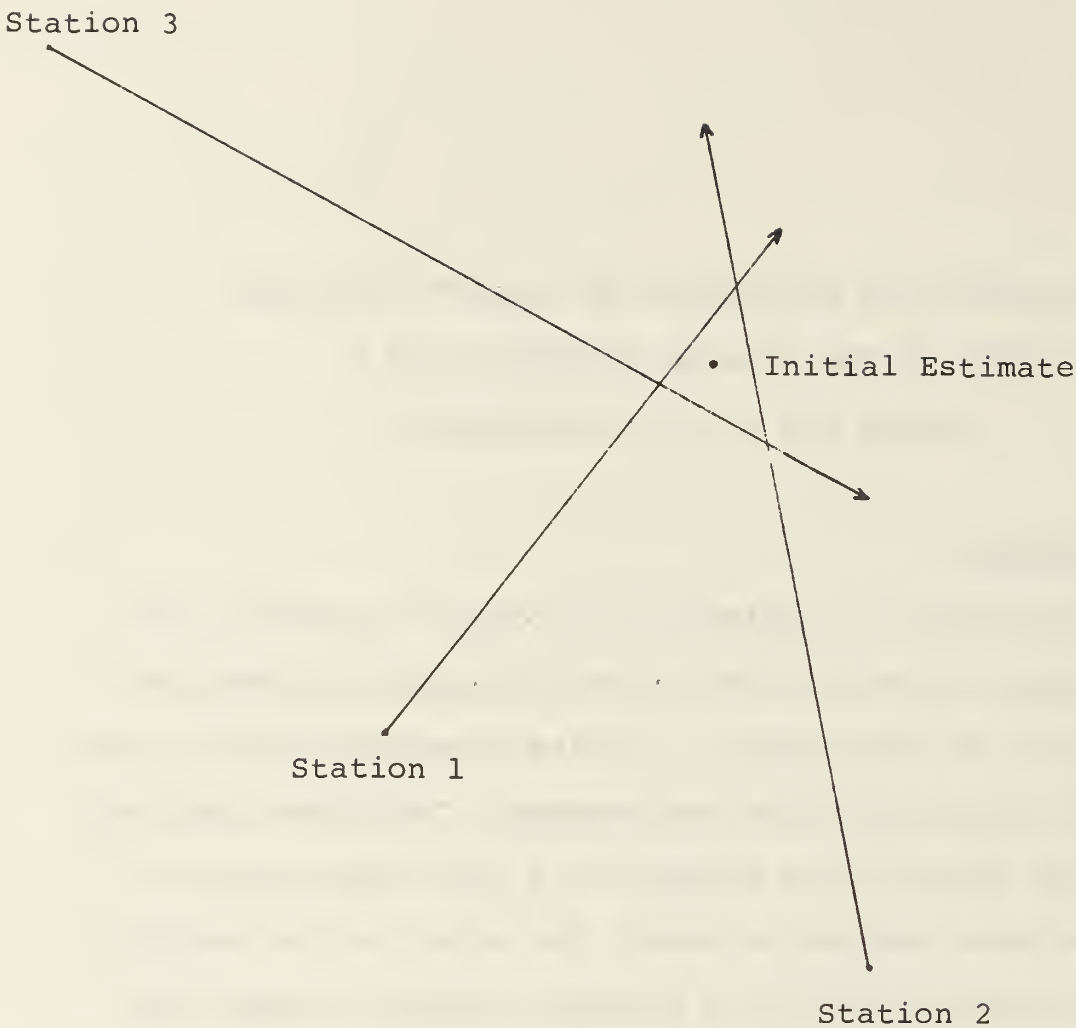


FIGURE 1. Geometry for the Example

Let the measured bearings and bearing errors (standard deviations) be:

$$\theta_1 = 35° \qquad e_1 = 4°$$

$$\theta_2 = 351° \qquad e_2 = 7°$$

$$\theta_3 = 131° \qquad e_3 = 5°$$

And let the ranges and bearings of the initial estimate be:

$$r_1 = 10{,}000 \text{ meters}, \qquad \beta_1 = 38°$$

$$r_2 = 15{,}000 \text{ meters}, \qquad \beta_2 = 346°$$

$$r_3 = 12{,}000 \text{ meters}, \qquad \beta_3 = 127° .$$

Use of the position estimation program with this data gives a final position estimate (fix) determined by:

$$x = -512 \text{ meters}$$

$$y = -\ 75 \text{ meters}$$

where x is its East-West distance and y is its North-South distance from the initial position estimate. The East-West, North-South xy-coordinate system with its origin at the initial estimate is shown in Figure 2. So the final position estimate is 512 meters to the West and 75 meters to the South of the initial position estimate.

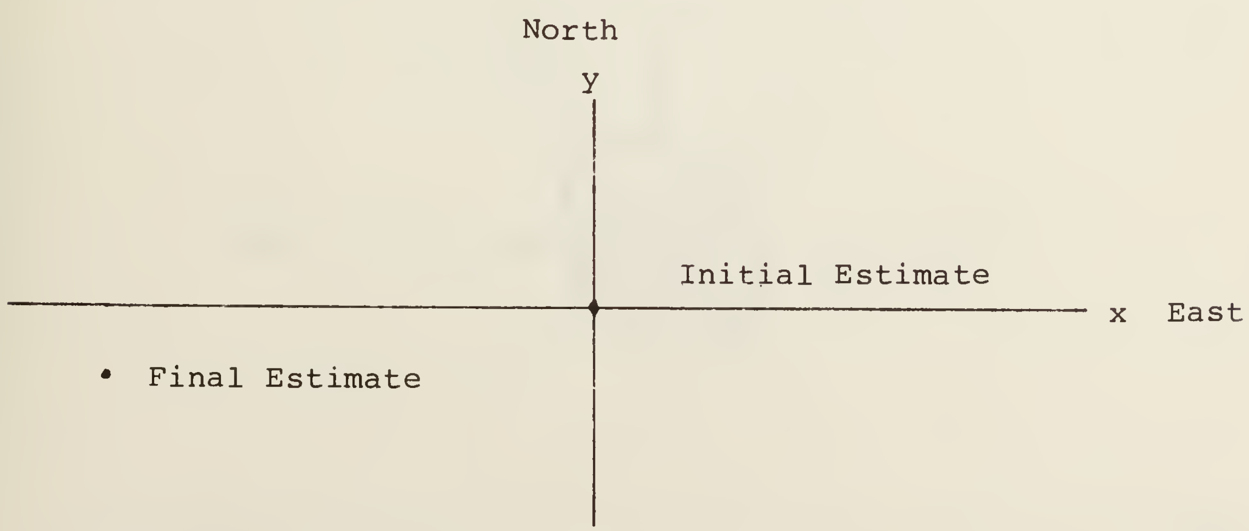


FIGURE 2. The Location of the Final Position Estimate with Respect to the Initial Position Estimate.

Minimum area elliptical confidence regions for an object's position can also be found by using the TI-59 program. The centers of the regions are at the fix, and their axes lie along the x' and y' axes of the coordinate system obtained by rotating the East-West, North-South xy-coordinate system with origin at the fix through an angle γ. The angle γ is defined so that it is positive for a rotation in the counterclockwise direction.

With the data from the above example, the program gives $\gamma = -31°$; so, the x' axis is directed 31° South of East. For a confidence region with minimum area and a confidence level of .9000, the ellipse bounding the region has a semi-major axis of 2064 meters, and a semi-minor axis of 1453 meters. The area of the region is 9.43 square killometers or 2.75 square nautical miles. The region is shown in Figure 3.

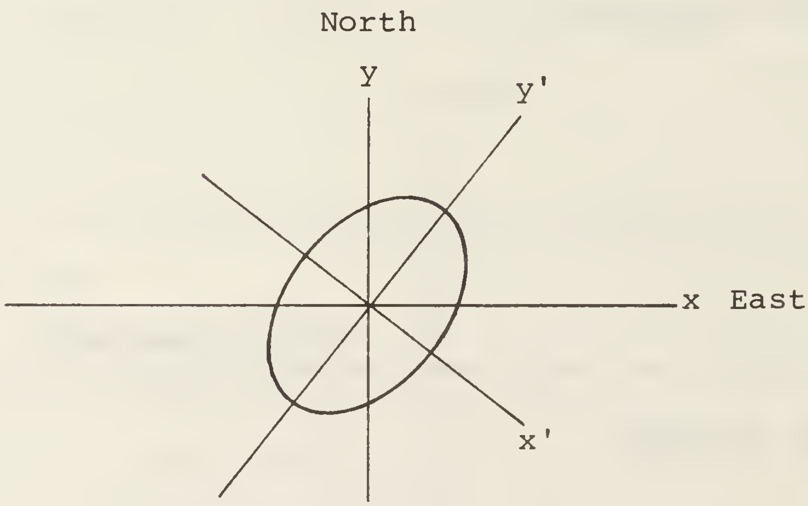


FIGURE 3. A .9000 Confidence Region for an Object's Position.

In the example discussed above, the position of the initial estimate is an input to the program. If this is not desirable, the program can be used to determine a position for the initial estimate. The position is the intersection of the two bearing lines corresponding to the first two bearings entered in the program. Both options are illustrated in Section II.

Since, in general, the smaller the bearing errors, the more likely that the initial estimate will be relatively near the object; small bearing errors can be considered to be a condition on the use of the procedure.

II. User Instructions

The TI-59 program to which the user instructions in this section apply can be used to calculate the quantities described in Section I.

The program requires the following inputs:

1. the observed bearing from or on an object for two or more stations;
2. station positions relative to a reference position; and
3. the bearing error (standard deviation) for each observed bearing.

Station positions can be specified in either of two ways. In the first way, Mode A, each station's position is specified in terms of its bearing α and its range ρ from a reference position. In the second way, Mode B, each station's position is specified in terms of its East-West distance x (plus for East) and its North-South distance (plus for North) from a reference position. The reference position can be any convenient location. For example, if it were at a station, then for that station $\alpha = 0$ and $\rho = 0$ or $x = 0$ and $y = 0$.

The program also requires an initial estimate of the object's position. The user has two options:

1. Let the program provide an estimate, or
2. Provide one with the input data.

For Option 1, the initial estimate is at the intersection of the bearing lines determined by the first two observed bearings entered into the program. For this reason, if this option is chosen, the first and second groups of data entered should correspond to the two stations estimated to have the smallest products $r_i e_i$. Although in this option the reference position cannot be at the initial estimate, it can be at one of the stations. If only two stations are involved, the final estimate is at the intersections of the bearing lines. (If the second option of either mode is used with an initial estimate which is not at the intersection of the two bearing lines, the coordinates of the final estimate will differ from coordinates of the intersection to the degree of the approximations involved in the estimation procedure.)

Two ways of providing confidence (probability) region data are available. In the first way, Mode C, the confidence (probability) p is specified. In the second way, Mode D, the multiplier k is specified where $k\sigma_{\hat{x}}$, and $k\sigma_{\hat{y}}$, are the semi-axes of the bounding ellipse.

The values of various quantities calculated by the program are either stored in registers or appear in the display. If a PC-100A printer is used, some of these values will be printed. The location of calculated values and the printing format is given after the user instructions. Those quantities which are not described in Section I are described below in the User Instructions or in Section IV.

All angles required or calculated by the program are in decimal degrees.

Step	Instructions	Enter	Press	Display
1.	If the calculator has been in use and flags have been set or the memory repartitioned, turn the calculator off and then on.			
2.	Read Side 1 and Side 2 of Card 1.			
3.	For confidence region calculations, Read Side 3 of Card 2.			
MODE A:	Station Locations Specified in Terms of Bearing and Range from a Reference Point.			
4a.	If the initial position estimate will be determined by the program, go to Step 7a. See the note on Page 10.			
5a.	Enter the initial estimate's bearing.	α^*	A'	
6a.	Enter the initial estimate's range.	ρ^*	R/S	
7a.	Enter the measured bearing on the object from a station or the reciprocal of the measured bearing on a station from the object.	θ_i	A	
8a.	Enter the station's bearing.	α_i	R/S	
9a.	Enter the station's range.	ρ_i	R/S	
10a.	Enter the bearing error.	e_i	R/S	i
11a.	Repeat Steps 7a, 8a, 9a and 10a for all stations. The number of repetitions i appears in the display after Step 10a.			

Step		Enter	Press	Display
MODE B:	Station Locations Specified in Terms of East-West Distance and North-South Distance from a Reference Point.			
4b.	If the initial position estimate will be determined by the program, go to Step 7b. See the note on Page 10.			
5b.	Enter the initial estimate's East-West distance.	x^*	B'	
6b.	Enter the initial estimate's North-South distance.	y^*	R/S	
7b.	Enter the measured bearing on the object from a station or the reciprocal of the measured bearing on a station from the object.	θ_i	B	
8b.	Enter the station's East-West distance.	x_i	R/S	
9b.	Enter the station's North-South distance.	y_i	R/S	
10b.	Enter the bearing error.	e_i	R/S	i
11b.	Repeat Steps 7b, 8b, 9b and 10b for all stations. The number of repetitions i appear in the display after Step 10b.			
BOTH MODES				
12.	Calculate the East-West distance, the North-South distance, the bearing and the range of the position estimate relative to the reference position. Also calculate the rotation angle γ, and the standard deviations $\sigma_{\hat{x}'}$ and $\sigma_{\hat{y}'}$. (See Section IV.)		R/S	

Step	Instructions	Enter	Press	Display
13.	For confidence (probability) region calculations, go to Step 14 if the confidence (probability) for the region is specified. If k is specified where $k\sigma_{\hat{x}'}$ and $k\sigma_{\hat{y}'}$ are the semi-axes of the bounding ellipse with the larger the major axis, go to Step 16.			
14.	Enter p, the confidence level (probability) and calculate k, $k\sigma_{\hat{x}'}$, $k\sigma_{\hat{y}'}$ and the area of the region. (The area units correspond to the distance units used.)	p	C	Area
15.	For a different value of p, go to Step 14.			
16.	Enter k and calculate the confidence level (probability) p, $k\sigma_{\hat{x}'}$, $k\sigma_{\hat{y}'}$ and the area of the region. (The area units correspond to the distance units used.)	k	D	Area
17.	For a different value of k, go to Step 16.			

NOTE: If a data entry error occurs in either mode, press RST and then use the following procedure: For Option 1, return to Step 7 and repeat all data entries. For Option 2, return to Step 5 and repeat all data entries.

Also, if a position estimate is to be determined for a new object position, follow this procedure.

NOTES:

a) The program printing format is given below:

For the initial data, i = 1, 2, ... , n with one space between groups:

Mode A		Mode B
α^*	initial estimate if provided	x^*
ρ^*		y^*
θ_i		θ_i
α_i		x_i
ρ_i		y_i
e_i		e_i

The format for the calculated position data is:

x

y

α

ρ

γ

$\sigma_{\hat{x}'}$

$\sigma_{\hat{y}'}$

For the confidence (probability) region portion of the program the format is after pressing either C or D:

p

k

$k\sigma_{\hat{x}'}$ semi-axis

$k\sigma_{\hat{y}'}$ semi-axis

Area

b) The following data is stored in the indicated registers:

Data	Registers
x^*	R38
y^*	R39
γ	R29
x	R30
y	R31
α	R32
ρ	R33
$\sigma_{\hat{x}'}$	R16
$\sigma_{\hat{y}'}$	R17
p	R14
k	R15
$k\sigma_{\hat{x}'}$	R18
$k\sigma_{\hat{y}'}$	R19

Four data tapes for a sample problem are given below. Distance units have not been specified, but they could be meters for example. Angles are in degrees. Option 1 (initial estimate not provided) for Mode A and Mode B is indicated by A and by B and Option 2 (initial estimate provided) is indicated by A' and B'.

For each mode and each option, the input data are indicated. The data determine the relative locations of three stations as well as the observed bearing of an object from each station.

For A and B, the reference location is at Station 1 and the initial position estimate (determined by the program) is at the intersection of the bearing lines for Station 1 and Station 2.

The intersection has coordinates $x^* = 906.4853528$ and $y^* = 17296.77092$ with respect to Station 1.

For A' and B', both the initial estimate and the reference location are at the intersection of the bearing lines, so $\alpha^* = 0$ and $\rho^* = 0$ and $x^* = 0$ and $y^* = 0$.

The data for A, B, A' and B' are all equivalent, and each solution gives the same data for a confidence (probability) region calculation. A tape with confidence (probability) region results for both Mode C and Mode D which correspond to A, B, A' and B' is given with the first four data tapes.

Following these data tapes, there is a data tape which illustrates the effects of using only the bearings for the first two stations. The tape is for Mode A, Option 1. The values stored in Registers 38 and 39 (the x and y coordinates of the intersection of the bearing lines from Station 1 and Station 2 with reference, in this case, to Station 1) are also listed on the tape (as well as given above.) And, as can be seen, the initial estimate and final estimate correspond.

A	
3.	θ_1
0.	α_1
0.	ρ_1
4.	e_1
33.	θ_2
273.	α_2
10000.	ρ_2
3.	e_2
303.	θ_3
33.	α_3
14000.	ρ_3
8.	e_3
573.5878933	x
16462.71223	y
1.995471725	α
16472.70157	ρ
-7.325392245	γ
787.3663755	$\sigma_{\hat{x}'}$
1233.080777	$\sigma_{\hat{y}'}$

A'	
0.	α^*
0.	ρ^*
3.	θ_1
183.	α_1
17320.50808	ρ_1
4.	e_1
33.	θ_2
213.	α_2
20000.	ρ_2
3.	e_2
303.	θ_3
129.5867755	α_3
8717.797886	ρ_3
8.	e_3
-332.8574567	x
-834.0586835	y
201.7584019	α
898.0393111	ρ
-7.325392259	γ
787.3663757	$\sigma_{\hat{x}'}$
1233.080776	$\sigma_{\hat{y}'}$

B

3.	θ_1
0.	x_1
0.	y_1
4.	e_1
33.	θ_2
-9986.295348	x_2
523.3595624	y_2
3.	e_2
303.	θ_3
7624.94649	x_3
11741.38795	y_3
8.	e_3
573.5878927	x
16462.71223	y
1.995471723	α
16472.70157	ρ
-7.325392245	γ
787.3663756	$\sigma_{\hat{x}'}$
1233.080777	$\sigma_{\hat{y}'}$

B'

0.	x^*
0.	y^*
3.	θ_1
-906.4853528	x_1
-17296.77092	y_1
4.	e_1
33.	θ_2
-10892.7807	x_2
-16773.41136	y_2
3.	e_2
303.	θ_3
6718.461137	x_3
-5555.382969	y_3
8.	e_3
-332.8974589	x
-834.0586906	y
201.7584019	α
898.0393184	ρ
-7.325392239	γ
787.3663755	$\sigma_{\hat{x}'}$
1233.080777	$\sigma_{\hat{y}'}$

C or D

0.9	p
2.145966026	k
1689.661493	$k\sigma_{\hat{x}'}$
2646.149454	$k\sigma_{\hat{y}'}$
14046364.97	Area
.8646647168	p
2.	k
1574.732751	$k\sigma_{\hat{x}'}$
2466.161553	$k\sigma_{\hat{y}'}$
12200517.6	Area

A

3.	θ_1	906.4853528	R38
0.	α_1	17296.77092	R39
0.	ρ_1		
4.	e_1		
33.	θ_2		
273.	α_2		
10000.	ρ_2		
3.	e_2		
906.4853528	x		
17296.77092	y		
3.	α		
17320.50808	ρ		
-20.35750198	γ		
818.886822	$\sigma_{\hat{x}'}$		
3092.663848	$\sigma_{\hat{y}'}$		
0.9	p		
2.145966026	k		
1757.303299	$k\sigma_{\hat{x}'}$		
6636.751549	$k\sigma_{\hat{y}'}$		
36639720.91	Area		
.8646647168	p		
2.	k		
1637.773644	$k\sigma_{\hat{x}'}$		
6185.327696	$k\sigma_{\hat{y}'}$		
31824857.22	Area		

To obtain the results given in Section I, use Mode A' and take the reference position at the initial estimate ($\alpha^* = 0$, $\rho^* = 0$). Then $\alpha_1 = 218°$, $\alpha_2 = 166°$ and $\alpha_3 = 307°$. The data tape for the calculation is given below.

A'

0.	α^*
0.	ρ^*
35.	θ_1
218.	α_1
10000.	ρ_1
4.	e_1
351.	θ_2
166.	$\alpha 2$
15000.	ρ_2
7.	e_2
131.	θ_3
307.	α_3
12000.	ρ_3
5.	e_3
-511.961856	x
-75.43753883	y
261.617789	α
517.4898687	ρ
-31.23492683	γ
677.2632305	$\sigma_{\hat{x}'}$
961.6888632	$\sigma_{\hat{y}'}$
0.9	p
2.145966026	k
1453.383883	$k\sigma_{\hat{x}'}$
2063.751628	$k\sigma_{\hat{y}'}$
9422966.381	Area
.864664716B	p
2.	k
1354.526461	$k\sigma_{\hat{x}'}$
1923.377726	$k\sigma_{\hat{y}'}$
8184684.605	Area

III. Program Listing

Before entering the program, press 2nd and then CP or turn the calculator off and then on. Next enter 5 in the display, press 2nd and then Op 17. This repartitions the calculator's memory so that the complete program can be entered. Two cards are required to record the complete program. The first card is for the position estimation portion of the program and for the first part of the confidence region portion of the program. The second card is for the second part of the confidence region program.

Before recording the program, enter 6 in the display, press 2nd and then Op 17. This returns the calculator's memory to the normal partition (479.59). Returning the calculator to the normal partition allows the two program cards to be read in the normal partition without forcing. When the program is used, it repartitions the calculator so that Bank 3 registers are program registers.

```
000  91  R/S
001  76  LBL
002  18  C'
003  69  OP
004  20   20
005  72  ST*
006  00   00
007  92  RTN
008  76  LBL
009  19  D'
010  69  OP
011  21   21
012  73  RC*
013  01   01
014  92  RTN
015  76  LBL
016  10  E'
017  65   ×
018  89   π
019  55   ÷
020  01   1
021  08   8
022  00   0
023  95   =
024  92  RTN
025  76  LBL
026  12   B
027  86  STF
028  01   01
029  76  LBL
030  11   A
031  32  X:T
032  87  IFF
033  00   00
034  00   00
035  56   56
036  05   5
037  69  OP
038  17   17
039  87  IFF
040  03   03
041  00   00
042  54   54
043  47  CMS
044  09   9
045  42  STO
046  00   00
047  42  STO
048  01   01
049  02   2
050  42  STO
051  09   09
052  69  OP
053  28   28
054  86  STF
055  00   00
056  32  X:T
057  99  PRT
058  18  C'
059  91  R/S
060  99  PRT
061  18  C'
062  91  R/S
063  99  PRT
064  18  C'
065  91  R/S
066  99  PRT
067  10  E'
068  18  C'
069  87  IFF
070  03   03
071  01   01
072  64   64
073  19  D'
074  87  IFF
075  01   01
076  00   00
077  88   88
078  75   -
079  19  D'
080  95   =
081  94  +/-
082  38  SIN
083  65   ×
084  19  D'
085  61  GTO
086  00   00
087  99   99
088  85   +
089  19  D'
090  32  X:T
091  19  D'
092  22  INV
093  37  P/R
094  24  CE
095  95   =
096  39  COS
097  65   ×
098  32  X:T
099  95   =
100  48  EXC
101  19   19
102  22  INV
103  97  DSZ
104  09   09
105  01   01
106  12   12
107  01   1
108  69  OP
109  21   21
110  98  ADV
111  91  R/S
112  42  STO
113  18   18
114  65   ×
115  43  RCL
116  14   14
117  42  STO
118  20   20
119  38  SIN
120  75   -
121  43  RCL
122  10   10
123  22  INV
124  44  SUM
125  20   20
126  38  SIN
127  65   ×
128  43  RCL
129  19   19
130  95   =
131  55   ÷
132  43  RCL
133  20   20
134  38  SIN
135  95   =
136  42  STO
137  38   38
138  43  RCL
139  14   14
140  39  COS
141  65   ×
142  43  RCL
143  18   18
144  75   -
145  43  RCL
146  10   10
147  39  COS
148  65   ×
149  43  RCL
```

```
150  19  19
151  95  =
152  55  ÷
153  43  RCL
154  20  20
155  38  SIN
156  95  =
157  42  STO
158  39  39
159  86  STF
160  03  03
161  02  2
162  42  STO
163  09  09
164  09  9
165  42  STO
166  00  00
167  42  STO
168  01  01
169  19  D'
170  10  E'
171  42  STO
172  18  18
173  19  D'
174  87  IFF
175  01  01
176  01  01
177  82  82
178  32  X:T
179  19  D'
180  32  X:T
181  37  P/R
182  75  -
183  43  RCL
184  38  38
185  95  =
186  94  +/-
187  32  X:T
188  22  INV
189  87  IFF
190  01  01
191  01  01
192  94  94
193  19  D'
194  75  -
195  43  RCL
196  39  39
197  95  =
198  94  +/-
199  32  X:T
200  22  INV
201  37  P/R
202  42  STO
203  19  19
204  10  E'
205  75  -
206  43  RCL
207  18  18
208  95  =
209  94  +/-
210  32  X:T
211  69  OP
212  21  21
213  64  PD*
214  01  01
215  42  STO
216  18  18
217  89  π
218  32  X:T
219  22  INV
220  77  GE
221  02  02
222  27  27
223  75  -
224  32  X:T
225  65  ×
226  02  2
227  95  =
228  49  PRD
229  18  18
230  73  RC*
231  01  01
232  35  1/X
233  42  STO
234  27  27
235  42  STO
236  26  26
237  32  X:T
238  43  RCL
239  19  19
240  37  P/R
241  42  STO
242  28  28
243  49  PRD
244  27  27
245  33  X²
246  44  SUM
247  23  23
248  43  RCL
249  18  18
250  49  PRD
251  26  26
252  49  PRD
253  27  27
254  32  X:T
255  49  PRD
256  28  28
257  49  PRD
258  26  26
259  33  X²
260  44  SUM
261  21  21
262  43  RCL
263  28  28
264  44  SUM
265  22  22
266  43  RCL
267  26  26
268  44  SUM
269  24  24
270  43  RCL
271  27  27
272  44  SUM
273  25  25
274  22  INV
275  97  DSZ
276  09  09
277  02  02
278  82  82
279  61  GTO
280  01  01
281  69  69
282  69  OP
283  28  28
284  43  RCL
285  08  08
286  98  ADV
287  91  R/S
288  43  RCL
289  23  23
290  42  STO
291  10  10
292  65  ×
293  43  RCL
294  21  21
295  22  INV
296  44  SUM
297  10  10
298  75  -
299  43  RCL
```

```
300  22   22
301  22  INV
302  49  PRD
303  10   10
304  33  X²
305  95   =
306  35  1/X
307  49  PRD
308  21   21
309  49  PRD
310  22   22
311  49  PRD
312  23   23
313  43  RCL
314  10   10
315  35  1/X
316  65   ×
317  02   2
318  95   =
319  22  INV
320  30  TAN
321  55   ÷
322  02   2
323  95   =
324  42  STO
325  29   29
326  43  RCL
327  23   23
328  65   ×
329  43  RCL
330  24   24
331  75   -
332  43  RCL
333  22   22
334  65   ×
335  43  RCL
336  25   25
337  85   +
338  43  RCL
339  38   38
340  95   =
341  42  STO
342  30   30
343  99  PRT
344  32  X:T
345  43  RCL
346  22   22
347  65   ×
348  43  RCL
349  24   24
350  75   -
351  43  RCL
352  21   21
353  65   ×
354  43  RCL
355  25   25
356  85   +
357  43  RCL
358  39   39
359  95   =
360  42  STO
361  31   31
362  99  PRT
363  32  X:T
364  22  INV
365  37  P/R
366  99  PRT
367  42  STO
368  32   32
369  01   1
370  32  X:T
371  99  PRT
372  42  STO
373  33   33
374  98  ADV
375  43  RCL
376  29   29
377  99  PRT
378  37  P/R
379  42  STO
380  11   11
381  33  X²
382  42  STO
383  12   12
384  42  STO
385  13   13
386  32  X:T
387  49  PRD
388  11   11
389  33  X²
390  42  STO
391  14   14
392  42  STO
393  15   15
394  43  RCL
395  21   21
396  49  PRD
397  12   12
398  49  PRD
399  14   14
400  43  RCL
401  22   22
402  65   ×
403  02   2
404  95   =
405  49  PRD
406  11   11
407  43  RCL
408  23   23
409  49  PRD
410  13   13
411  49  PRD
412  15   15
413  43  RCL
414  15   15
415  85   +
416  43  RCL
417  11   11
418  85   +
419  43  RCL
420  12   12
421  95   =
422  34  √X
423  99  PRT
424  42  STO
425  16   16
426  43  RCL
427  13   13
428  75   -
429  43  RCL
430  11   11
431  85   +
432  43  RCL
433  14   14
434  95   =
435  34  √X
436  99  PRT
437  42  STO
438  17   17
439  81  RST
440  76  LBL
441  17  B'
442  86  STF
443  02   02
444  76  LBL
445  16  A'
446  99  PRT
447  32  X:T
448  47  CMS
449  09   9
```

```
450  42  STO
451  00   00
452  42  STO
453  01   01
454  91  R/S
455  99  PRT
456  87  IFF
457  02   02
458  04   04
459  63   63
460  32  X:T
461  37  P/R
462  32  X:T
463  42  STO
464  39   39
465  32  X:T
466  42  STO
467  38   38
468  86  STF
469  03   03
470  98  ADV
471  91  R/S
472  76  LBL
473  14   D
474  86  STF
475  04   04
476  76  LBL
477  13   C
478  98  ADV
479  87  IFF
480  04   04
481  05   05
482  22   22
483  42  STO
484  14   14
485  99  PRT
486  75   -
487  01   1
488  95   =
489  94  +/-
490  23  LNX
491  65   ×
492  02   2
493  95   =
494  94  +/-
495  34  √X
496  42  STO
497  15   15
498  99  PRT
499  65   ×
500  43  RCL
501  16   16
502  95   =
503  99  PRT
504  42  STO
505  18   18
506  65   ×
507  53   (
508  43  RCL
509  15   15
510  65   ×
511  43  RCL
512  17   17
513  54   )
514  99  PRT
515  42  STO
516  19   19
517  65   ×
518  89   π
519  95   =
520  99  PRT
521  81  RST
522  42  STO
523  15   15
524  33  X²
525  55   ÷
526  02   2
527  95   =
528  94  +/-
529  22  INV
530  23  LNX
531  75   -
532  01   1
533  95   =
534  94  +/-
535  42  STO
536  14   14
537  99  PRT
538  43  RCL
539  15   15
540  61  GTO
541  04   04
542  98   98
543  00   0
544  00   0
545  00   0
546  00   0
547  00   0
548  00   0
549  00   0
550  00   0
551  00   0
552  00   0
553  00   0
554  00   0
555  00   0
556  00   0
557  00   0
558  00   0
559  00   0
```

IV. A Development for the Procedure

In the development for the estimation procedure given here, all angles are in radians and the assumptions stated in Section I apply.

Figure 4 shows three bearing lines from the ith of n stations. One is the observed bearing line of an object. One of length ρ_i goes to the origin of an xy-coordinate system located at the object's unknown position. And one of length r_i goes to an initial estimate with known position but unknown coordinates (x,y). Note, estimates for $-x$ and $-y$ estimate the object's position. To find estimates $-\hat{x}$ and $-\hat{y}$, consider the arc coordinates $u_i = \rho_i(\theta_i - \phi_i)$ of the observed bearing line and $v_i = \rho_i(\beta_i - \phi_i)$ of the bearing line to the point (x,y). They are defined by the three bearing lines and the circle of radius ρ_i which goes through the object's position and which is centered on the station as shown in Figure 4.

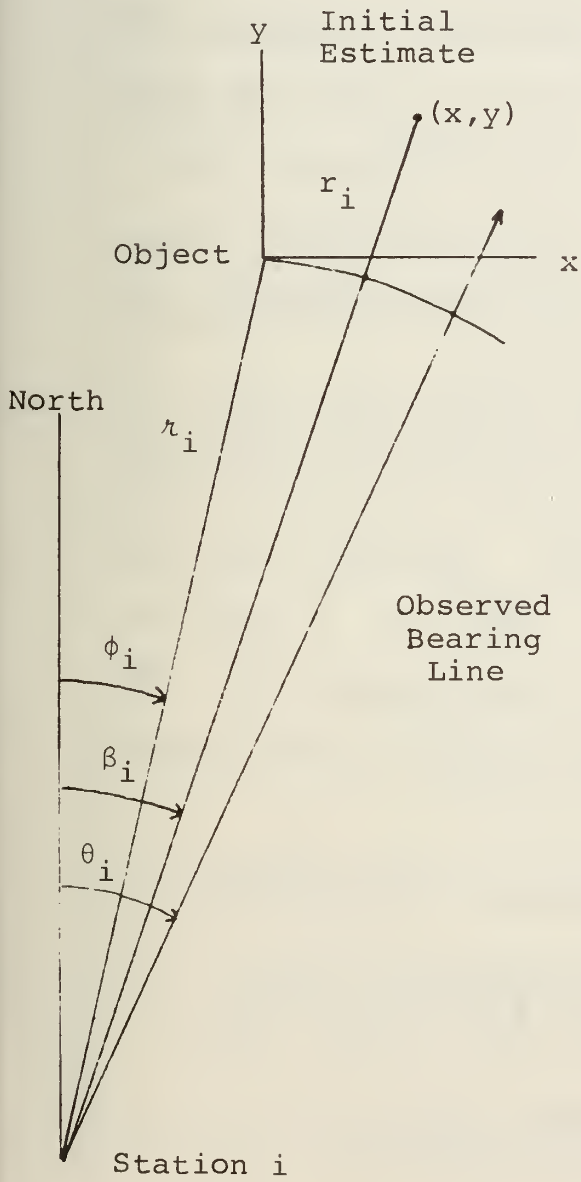


FIGURE 4. Problem Geometry.

By defining $w_i = r_i(\theta_i - \beta_i)$ (all angles in radians), $u_i = v_i + w_i$. Note, $\theta_i - \beta_i$ is known, but $\beta_i - \phi_i$ is not. However, v_i can be expressed in terms of x and y, and, to first order, $v_i = x \cos \beta_i - y \sin \beta_i$; so, if $\tan(\beta_i - \phi_i) \simeq (\beta_i - \phi_i)$ for $i = 1,2,\ldots,n$, that is, if (x,y) is relatively near the object's position, $u_i \simeq r_i(\theta_i - \beta_i) + x \cos \beta_i - y \sin \beta_i$ for $i = 1,2,\ldots,n$.

In this development, the initial estimate is assumed to be relatively close enough to the object's position so that the above approximation for u_i can be used and so that r_i can be taken equal to r_i. With this assumption, u_i is a function of: θ_i, the observed value of a random quantity; the known parameters r_i and β_i; and the unknown parameters x and y.

If a distribution for the Θ_i can be specified, then a distribution for the U_i can be determined and standard estimates $\hat{x}$ and $\hat{y}$ for x and y can be considered. Maximum likelihood estimates are discussed in this section. Each Θ_i is taken to be a normal random variable with mean ϕ_i and variance e_i^2. And the n random variables Θ_i, $i = 1,2,\ldots,n$ (one for each station) are taken to be independent.

The likelihood for a sample $\theta_1, \theta_2, \ldots, \theta_n$ is then

$$L(\theta_1, \theta_2, \ldots, \theta_n) = \prod_1^n \left(\frac{1}{\sqrt{2\pi}\, e_i}\right) \exp\left[-\frac{1}{2}\sum_1^n (\theta_i - \phi_i)^2/e_i^2\right]$$

and the likelihood for a corresponding sample $u_1, u_2, \ldots, u_n$ is

$$L(u_1,u_2,\ldots,u_n) = \prod_{i=1}^{n} \left(\frac{1}{\sqrt{2\pi}\,\sigma_i}\right) \exp\left[-\frac{1}{2}\sum_{1}^{n} u_i^2/\sigma_i^2\right]$$

where $\sigma_i = r_i e_i$ (with e_i in radians) since $u_i = r_i(\theta_i - \phi_i)$.

By definition, the maximum likelihood estimates of x and y are the estimates $\hat{x}$ and $\hat{y}$ which make $L(u_1,u_2,\ldots,u_n)$ a maximum. In this case, making $L(u_1,u_2,\ldots,u_n)$ a maximum is equivalent to making $\sum_1^n (u_i^2/\sigma_i^2)$ a minimum. So, to find $\hat{x}$ and $\hat{y}$, solve the following two equations for x and y:

$$\frac{\partial(\ell n\ L)}{\partial x} = 0 \quad \text{and} \quad \frac{\partial(\ell n\ L)}{\partial y} = 0 \ .$$

The solutions are $x = \hat{x}$ and $y = \hat{y}$, and $\hat{x}$ and $\hat{y}$ are the maximum likelihood estimates. With $w_i = r_i(\theta_i - \beta_i)$ and the conditions assumed above these two equations are linear equations in x and y. And,

$$\sum_{1}^{n} [w_i + \hat{x}\cos\beta_i - \hat{y}\sin\beta_i](\cos\beta_i)/\sigma_i^2 = 0$$

and

$$\sum_{1}^{n} [w_i + \hat{x}\cos\beta_i - \hat{y}\sin\beta_i](\sin\beta_i)/\sigma_i^2 = 0 \ .$$

And, in terms of the following quantities:

$$A = \Sigma(\cos^2\beta_i)/\sigma_i^2 \ , \qquad B = \Sigma(\sin\beta_i\cos\beta_i)/\sigma_i^2 \ ,$$

$$C = \Sigma(\sin^2\beta_i)/\sigma_i^2 \ , \qquad D = \Sigma(w_i\cos\beta_i)/\sigma_i^2 \ ,$$

$$E = \Sigma(w_i\sin\beta_i)/\sigma_i^2 \ ,$$

the equations are:

$$A\hat{x} - B\hat{y} = -D$$

$$B\hat{x} - C\hat{y} = -E \ .$$

So the solutions are:

$$\hat{x} = (BE - CD)/(AC - B^2)$$

$$\hat{y} = (AE - BD)/(AC - B^2) \ .$$

A confidence region can be constructed about an estimated position. In order to indicate how this is done, a probability region about the true position will be considered first.

Note, $\hat{x}$ and $\hat{y}$ are values of random variables. If a new set of bearings $\theta_1, \theta_2, \ldots, \theta_n$ is observed (for a fixed initial estimate and object), in general, a new pair of values $\hat{x}$ and $\hat{y}$ will be obtained.

If $\hat{X}$ and $\hat{Y}$ represent these random variables, then

$$\hat{X} = \frac{1}{(AC-B^2)} \sum_{1}^{n} (W_i/\sigma_i^2)(B \sin \beta_i - C \cos \beta_i)$$

$$\hat{Y} = \frac{1}{(AC-B^2)} \sum_{1}^{n} (W_i/\sigma_i^2)(A \sin \beta_i - B \cos \beta_i)$$

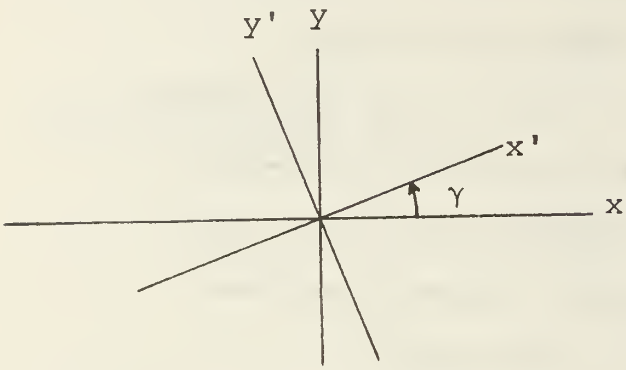
with $W_i = r_i(\Theta_i - \beta_i)$. ($W_i$ is the random distance intercepted along the ith arc between the bearing lines defined by Θ_i and β_i.)

Note, $\hat{X}$ and $\hat{Y}$ have a bivariate normal distribution, since they are a linear combination of the n normal random variables $W_1, W_2, \ldots, W_n$, or equivalently of the n normal random variables $\Theta_1, \Theta_2, \ldots, \Theta_n$. Also $E(W_i) = r_i(\phi_i - \beta_i)$.

If $\beta_i = \phi_i$ for $i = 1,2,\ldots,n$, that is, if the initial estimate of the object's position is at the object's position, $E(W_1) = 0$ for $i = 1,2,\ldots,n$. And, therefore, $E(\hat{X}) = 0$ and $E(\hat{Y}) = 0$. So, in this case, the "location" of the bivariate normal distribution of a point $(\hat{X}, \hat{Y})$, the random coordinates of the object's estimated position, is the same as that for the point $(-\hat{X}, -\hat{Y})$ and both are centered on the object's position. However, the "location" of the distribution of $(-\hat{X},-\hat{Y})$ is independent of the location of the initial estimate when the coordinates $(-\hat{X},-\hat{Y})$ refer to a coordinate system with origin at the initial estimate. This fact simplifies the establishment of a confidence region about the location of an estimated position.

A region of minimum area for a given probability of containment of an estimated position can be determined. The region is bounded by an ellipse which is centered on the object's position and whose axes lie along the axes of an x'y'-coordinate system obtained by rotating the xy-coordinate system centered on the object's position through an angle γ. In this system, $\sigma_{\hat{x}\hat{y}} = 0$, that is, $\hat{X}'$ and $\hat{Y}'$ are independent normal random variables.

The two coordinate systems are illustrated in Figure 5. The coordinates of a point in the two systems are related by



$$x' = x \cos\gamma + y \sin\gamma$$

$$y' = -x \sin\gamma + y \cos\gamma$$

These relations, along with $\sigma_{\hat{x}'\hat{y}'} = 0$, imply:

FIGURE 5. Rotation Geometry.

$$\sigma^2_{\hat{x}'} = \sigma^2_{\hat{x}} \cos^2\gamma + 2\sigma_{\hat{x}\hat{y}} \cos\gamma \sin\gamma + \sigma^2_{\hat{y}} \sin^2\gamma \ ,$$

$$\sigma^2_{\hat{y}'} = \sigma^2_{\hat{x}} \sin^2\gamma - 2\sigma_{\hat{x}\hat{y}} \cos\gamma \sin\gamma + \sigma^2_{\hat{y}} \cos^2\gamma$$

and

$$\tan 2\gamma = \frac{2\sigma_{\hat{x}\hat{y}}}{\sigma^2_{\hat{x}} - \sigma^2_{\hat{y}}}$$

where γ, the angle of rotation of the coordinate axes, is positive in the counterclockwise direction.

With the initial estimate of the object's position at the object's position ($\beta_i = \phi_i$, $i = 1,2,\ldots,n$), so $E(W_i) = 0$ and $\mathrm{Var}(W_i) = \sigma_i^2$,

$$\sigma^2_{\hat{x}} = \frac{1}{(AC-B^2)^2} \sum_1^n (1/\sigma_i^2)(B \sin \beta_i - C \cos \beta_i)^2,$$

$$\sigma^2_{\hat{y}} = \frac{1}{(AC-B^2)^2} \sum_1^n (1/\sigma_i^2)(A \sin \beta_i - B \cos \beta_i)^2$$

and

$$\sigma_{\hat{x}\hat{y}} = \frac{1}{(AC-B^2)^2} \sum_1^n (1/\sigma_i^2)(B \sin \beta_i - C \cos \beta_i)(A \sin \beta_i - B \cos \beta_i)$$

Using the definition for A, B and C, the above become

$$\sigma^2_{\hat{x}} = \frac{C}{(AC-B^2)},$$

$$\sigma^2_{\hat{y}} = \frac{A}{(AC-B^2)},$$

and

$$\sigma_{\hat{x}\hat{y}} = \frac{B}{(AC-B^2)}.$$

So, $\tan 2\gamma = 2B/(C-A)$ for $\beta_i = \phi_i$, $i = 1,2,\ldots,n$.

With the object's position known and, hence, ϕ_i known for $i = 1,2,\ldots,n$, the above equations for $\sigma^2_{\hat{x}}$, $\sigma^2_{\hat{y}}$, $\sigma_{\hat{x}\hat{y}}$ and γ can be used, since the initial estimate of the object's position can be taken as the object's position.

With values for $\sigma_{\hat{x}}$, $\sigma_{\hat{y}}$, $\sigma_{\hat{x}\hat{y}}$ and γ, values for $\sigma_{\hat{x}'}$ and $\sigma_{\hat{y}'}$ can be found by using the equations in the middle of Page 30. And then, the probability that an estimated position will be within an ellipse of semiaxes $k\sigma_{\hat{x}'}$ and $k\sigma_{\hat{y}'}$

which is centered on the object's position can be found. It is $1 - \exp(-k^2/2)$. (This result follows from integrating the bivariate normal density over the ellipse.) And the area of the ellipse is $\pi k^2 \sigma_{\hat{x}'} \sigma_{\hat{y}'}$.

Given estimates $\hat{x}$ and $\hat{y}$ found by using the relations on Page 28, an ellipse with semi-axes $k\sigma_{\hat{x}'}$ and $k\sigma_{\hat{y}'}$ centered on the point with coordinates $(-\hat{x},-\hat{y})$ in a coordinate system with origin at the initial estimate and oriented as indicated by γ is a $1 - \exp(-k^2/2)$ confidence region. This follows from the bivariate normal distribution of $-\hat{X}$ and $-\hat{Y}$ which in this system is centered on the object's position. The ellipse is defined if $\sigma^2_{\hat{x}}$, $\sigma^2_{\hat{y}}$ and $\sigma_{\hat{x}\hat{y}}$ are known (the covariance matrix is known). And to the degree of the approximations involved, this can be assumed to be the case. In particular, by assuming the initial estimate of the object's position is at the object's position, which is consistent with assuming $(\beta_i - \phi_i)$ is small, values for $\sigma^2_{\hat{x}}$, $\sigma^2_{\hat{y}}$, $\sigma_{\hat{x}\hat{y}}$ and γ can be obtained by using the relations on Page 31. These values can then be used to determine $\sigma^2_{\hat{x}'}$ and $\sigma^2_{\hat{y}'}$ by using the relations on Page 30. And, then, with a value for k, a confidence region can be constructed. To the degree of the approximations involved, the shape of the confidence region is independent of both the object's position and of the initial estimate of the object's position.

For the case where bearings are taken from the object on two or more stations, θ_i is the reciprocal of the bearing taken from the object.

A discussion for this and for other bearings only position estimation procedures for situations similar to the one considered here is given in Reference 1 listed below. Reference 2 gives an equivalent bearings only procedure. It also gives a range only procedure, a range and bearing procedure and HP-9830A programs with which to implement the procedures. Using the fix determined by two lines of bearing as the initial estimate was suggested by this reference.

The equations used in the program to determine (x^*, y^*), the coordinates of the fix, are:

$$x^* \sin(\theta_2-\theta_1) = [\rho_1 \sin(\alpha_1-\theta_1)] \sin\theta_2 - [\rho_2 \sin(\alpha_2-\theta_2)] \sin\theta_1$$

$$y^* \sin(\theta_2-\theta_1) = [\rho_1 \sin(\alpha_1-\theta_1)] \cos\theta_2 - [\rho_2 \sin(\alpha_2-\theta_2)] \cos\theta_1$$

References:

1. Schrader, John Yale, Jr., "An Alternative Approach to Long Range DF Fixing," Naval Postgraduate School Ph.D. Thesis, September 1974.

2. Thompson, K.P. and Kullback, J.H., "Position-Fixing and Position-Predicting Programs for the Hewlett-Packard Model 9830A Programmable Calculator," NRL Memorandum Report 3265, Naval Research Laborator Washington, D.C.

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